

INTERPOLATION AND SAMPLING IN SMALL BERGMAN SPACES

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ABSTRACT. Carleson measures and interpolating and sampling sequences for weighted Bergman spaces on the unit disk are described for weights that are radial and grow faster than the standard weights $(1 - |z|)^{-\alpha}$, $0 < \alpha < 1$. These results make the Hardy space H^2 appear naturally as a “degenerate” endpoint case for the class of Bergman spaces under study.

1. INTRODUCTION

This work originates in my 1993 paper [5] which concerns interpolation and sampling in Bergman spaces on the unit disk with standard radial weights $(1 - |z|^2)^{-\alpha}$ and $\alpha < 1$. Following [5], A. Borichev, R. Dhuez, and K. Kellay [1] studied the same problem when the weights decay more rapidly than any positive power of $1 - |z|$ as $|z| \nearrow 1$. What remains to be settled is then the case of nontrivial weights growing more rapidly than $(1 - |z|)^{-\alpha}$ for any α in $(0, 1)$, which should be thought of as dealing with Hilbert spaces of analytic functions lying “between” the classical Hardy and Bergman spaces. In what follows, I will show how this can be done. Somewhat vaguely phrased, the present analysis offers a “smooth” transition from the Hardy space situation and L. Carleson’s theorems (in this context a “degenerate” endpoint case) and the setting of Bergman spaces with standard weights.

Throughout this paper w will be a positive and continuous function on $[0, 1)$, fixed once and for all, such that for a positive constant c

$$(1) \quad w(1 - t) \geq cw(1 - 2t)$$

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whenever $0 < t \leq 1/2$. We will assume that w is integrable and for convenience that

$$\int_0^1 w(x)dx = 1.$$

With our fixed weight w , we associate the weighted Bergman space A_w^2 consisting of all functions f analytic in the open unit disk \mathbb{D} satisfying

$$\int_{z \in \mathbb{D}} |f(z)|^2 w(|z|) d\sigma(z) < \infty,$$

where σ denotes Lebesgue area measure on \mathbb{D} . The latter integral defines a norm on A_w^2 , but we prefer to use another equivalent norm. Define $0 \leq r_n < 1$ by the relation

$$\int_{r_n}^1 w(x)dx = 2^{-n}$$

for every nonnegative integer n , and set

$$(2) \quad \|f\|_w^2 = \sum_{n=1}^{\infty} 2^{-n} \int_0^{2\pi} |f(r_n e^{it})|^2 \frac{dt}{2\pi}.$$

If we had chosen to start from the sequence (r_n) instead of the weight w , then we would have needed to replace the condition (1) by the requirement that

$$(3) \quad \inf_{n \geq 0} \frac{1 - r_n}{1 - r_{n+1}} > 1.$$

This alternative approach has the advantage that it permits us to associate the Hardy space H^2 of the unit disk with the “degenerate” case when the sequence of radii r_n is allowed to be finite and $\max_n r_n = 1$.

To see how the scale of Bergman spaces with standard weights fits into this context, we introduce the following scale of weights associated with w :

$$w_\alpha(x) = (1 - \alpha)w(x) \left(\int_x^1 w(t)dt \right)^{-\alpha}$$

for $\alpha < 1$. It is plain that we also have $w_\alpha(1 - t) \geq c_\alpha w_\alpha(1 - 2t)$ for some constant c_α , and that $\int_0^1 w_\alpha(x)dx = 1$. If we choose $w \equiv 1$, then the family of weights w_α corresponds to the standard weighted Bergman spaces. Note that substituting w by w_α corresponds to replacing 2^{-n} in (2) by $2^{-(1-\alpha)n}$. It may be verified that this implies that the Carleson measures are described in the same way for all the spaces $A_{w_\alpha}^2$ and that the notion of density that we will use for A_w^2 , also applies to describe interpolating and sampling sequences for each of the spaces $A_{w_\alpha}^2$.

The next sections contain two theorems, the first describing the Carleson measures for our space A_w^2 and the second the interpolating and sampling sequences for the same space.

The first theorem is easily proved using Carleson's embedding theorem, while the second requires somewhat delicate technicalities. A main ingredient in the proof of the second theorem is a lemma involving a method of redistribution and atomization of certain Riesz measures.

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2. CARLESON MEASURES FOR A_w^2

Given a Hilbert space \mathcal{H} of analytic functions on \mathbb{D} , we say that a nonnegative Borel measure μ on \mathbb{D} is a Carleson measure for \mathcal{H} if there exists a positive constant C such that

$$\int_{\mathbb{D}} |f(z)|^2 d\mu(z) \leq C \|f\|_{\mathcal{H}}^2$$

holds for every f in \mathcal{H} . The Carleson constant of μ is the smallest possible constant C for which this holds. In our case, \mathcal{H} will be either A_w^2 or the Hardy space H^2 , where the latter consists of all analytic functions f in \mathbb{D} for which

$$\|f\|_{H^2} = \sup_{r < 1} \int_0^{2\pi} |f(re^{it})|^2 \frac{dt}{2\pi} < \infty.$$

A classical theorem of L. Carleson [3] says that μ is a Carleson measure for H^2 if and only if there is a positive constant C such that we have $\mu(Q_\zeta) \leq C(1 - |\zeta|)$ for every Carleson "square" $Q_\zeta = \{z : |\zeta| < |z| < 1, \arg(z\bar{\zeta}) < 1 - |\zeta|\}$, i.e., for every point ζ in $\mathbb{D} \setminus \{0\}$.

Before stating our theorem on Carleson measures, we introduce the following notations, to be retained for the remainder of this paper. Set

$$\Omega_n = \{z : r_n \leq |z| < r_{n+1}\},$$

and let μ_n be the measure such that $d\mu_n(z) = \chi_{\Omega_n}(z)d\mu(z)$ whenever a nonnegative Borel measure μ on \mathbb{D} is given. The notation $U(z) \lesssim V(z)$ (or equivalently $V(z) \gtrsim U(z)$) means that there is a constant C such that $U(z) \leq CV(z)$ holds for all z in the set in question, which may be a space of functions or a set of numbers. If both $U(z) \lesssim V(z)$ and $V(z) \lesssim U(z)$, then we write $U(z) \simeq V(z)$.

Theorem 1. *A nonnegative Borel measure μ on \mathbb{D} is a Carleson measure for A_w^2 if and only if each μ_n is a Carleson measure for H^2 with Carleson constant $\lesssim 2^{-n}$.*

Proof. The proof relies on Carleson's theorem [3]. For the necessity, it suffices to check Carleson "squares" $Q_\zeta = \{z : |\zeta| < |z| < 1, \arg(z\bar{\zeta}) < 1 - |\zeta|\}$ whose top center ζ is in Ω_n . We use the test function $f_\zeta(z) = (1 - \bar{\zeta}z)^{-\gamma}$ with γ so large that

$$\|f_\zeta\|_w^2 \simeq 2^{-n}(1 - |\zeta|)^{-2\gamma+1};$$

this can be achieved because of (3). It follows readily from the Carleson measure condition that $\mu(Q_\zeta \cap \Omega_n) \lesssim 2^{-n}(1 - |\zeta|)$ as required by Carleson's theorem.

To prove the sufficiency, we note that if μ_n is a Carleson measure for H^2 with Carleson constant $\lesssim 2^{-n}$, then, in view of (3), the same holds for H^2 of the smaller disk $r_{n+2}\mathbb{D}$. Given an arbitrary function f in A_w^2 , we sum the corresponding Carleson measure estimates over n and get

$$\int_{\mathbb{D}} |f(z)|^2 d\mu(z) \lesssim \sum_{n=0}^{\infty} 2^{-n} \int_0^{2\pi} |f(r_{n+2}e^{it})|^2 \frac{dt}{2\pi}.$$

□

3. INTERPOLATION AND SAMPLING IN A_w^2

Let \mathcal{H} be as in the previous section, and let K_z be the reproducing kernel for \mathcal{H} at the point z in \mathbb{D} . We say that a sequence $\Lambda = (\lambda_j)$ of distinct points in \mathbb{D} is an interpolating sequence for \mathcal{H} if we can solve the interpolation problem $f(\lambda_j) = a_j$ whenever the sequence (a_j) satisfies the admissibility condition

$$\sum_j \frac{|a_j|^2}{K_{\lambda_j}(\lambda_j)} < \infty;$$

the sequence Λ is said to be a sampling sequence if there are positive constants A and B such that

$$(4) \quad A\|f\|_{\mathcal{H}}^2 \leq \sum_j \frac{|f(\lambda_j)|^2}{K_{\lambda_j}(\lambda_j)} \leq B\|f\|_{\mathcal{H}}^2$$

for every f in \mathcal{H} .

We are interested in such sequences when $\mathcal{H} = A_w^2$, and we therefore need a precise estimate for $K_z(z)$ in this case. To this end, we recall that $K_z(z)$ is the square of the norm of the functional of point evaluation $f \mapsto f(z)$ on A_w^2 . By (3), we have that $1 - |z| \leq c(r_{n+2} - |z|)$ when z is in Ω_n for some constant c independent of n . Thus

$$(5) \quad |f(z)|^2 \leq C2^n(1 - |z|)^{-1}\|f\|_w^2$$

for every f in A_w^2 and z in Ω_n with C independent of n . On the other hand, choosing f_z as in the proof of Theorem 1, we get that

$$|f_z(z)|^2 \gtrsim 2^n(1 - |z|)^{-1} \|f_z\|_w^2$$

if γ is again chosen sufficiently large; we conclude that

$$(6) \quad K_z(z) \simeq 2^n(1 - |z|)^{-1}$$

for z in Ω_n and all n when K_z is the reproducing kernel for A_w^2 .

We denote by $\varrho(z, \zeta)$ the pseudohyperbolic distance between two points z and ζ in \mathbb{D} , i.e.,

$$\varrho(z, \zeta) = \left| \frac{z - \zeta}{1 - \bar{\zeta}z} \right|.$$

Let $\Lambda = (\lambda_j)$ be a separated sequence in \mathbb{D} , which as usual we take to mean that $\inf_{j \neq l} \varrho(\lambda_j, \lambda_l) > 0$. For a given z , let $n(z) = n(|z|)$ be the nonnegative integer such that $r_{n(z)} \leq |z| < r_{n(z)+1}$. We then define the following densities:

$$D_w^+(\Lambda) = \limsup_{m \rightarrow \infty} \frac{1}{m} \sup_{|z| < 1} \sum_{|\lambda_j| \leq r_{n(z)+m}} (1 - \varrho(z, \lambda_j))$$

and

$$D_w^-(\Lambda) = \liminf_{m \rightarrow \infty} \frac{1}{m} \inf_{|z| < 1} \sum_{|\lambda_j| < r_{n(z)+m}} (1 - \varrho(z, \lambda_j)).$$

We future reference, we record the following consequence of Theorem 1.

Lemma 1. *The measure*

$$\mu = \sum_{n=0}^{\infty} 2^{-n} \sum_{r_n < |\lambda_j| < r_{n+1}} (1 - |\lambda_j|) \delta_{\lambda_j}$$

μ is a Carleson measure for A_w^2 if and only if $D_w^+(\Lambda) < \infty$.

Our main result is the following theorem.

Theorem 2. (I) A sequence Λ is an interpolating sequence for A_w^2 if and only if it is separated and $D_w^+(\Lambda) < (\log 2)/2$. (S) A separated sequence Λ is a sampling sequence for A_w^2 if and only if $D_w^+(\Lambda) < \infty$ and $D_w^-(\Lambda) > (\log 2)/2$.

In the “degenerate” case of H^2 (when the sequence of radii r_n is allowed to be finite and $\max_n r_n = 1$), H. Shapiro and A. Shield’s L^2 version of Carleson’s interpolation theorem

[7, 2] gives that the condition $D_w^+(\Lambda) < (\log 2)/2$ in part (I) should be replaced by the simpler condition

$$\sup_{|z|<1} \sum_j (1 - \varrho(z, \lambda_j)) < \infty;$$

it is well-known that there is no counterpart to part (S) when A_w^2 is replaced by H^2 .

The densities used in Theorem 2 are defined somewhat differently from those used in the original paper [5] and in [6]. These densities can also be defined via harmonic measure as shown in [4]. It seems clear that our Theorem 2 can be rephrased in a similar way using harmonic measure. One can of course also prove similar results for Bergman L^p spaces without any essential changes of the arguments.

The remainder of this paper is devoted to proving respectively the necessity (Section 4) and the sufficiency (Section 5) of the conditions of Theorem 2.

4. PROOF OF THE NECESSITY OF THE CONDITIONS OF THEOREM 2

We begin with part (I). To see that an interpolating sequence is separated, we can argue similarly as in the proof of Theorem 1. Namely, if Λ is an interpolating sequence for A_w^2 , then the sequence $\Lambda \cap \Omega_n$ is an interpolating sequence for H^2 of the smaller disk $r_{n+2}\mathbb{D}$, and then, in view of (3), we may argue as we do in the classical H^2 case. We omit the details of this routine argument. Lemma 1 gives that $D_w^+(\lambda) < \infty$ holds when Λ is an interpolating sequence.

Just as on pages 57–58 in [6] we may modify the definition of the upper density:

$$D_w^+(\Lambda) = \limsup_{m \rightarrow \infty} \frac{1}{m} \sup_{\lambda_l} \sum_{|\lambda_j| \leq r_{n(\lambda_l)+m}} (1 - \varrho(\lambda_l, \lambda_j)).$$

We also repeat the argument on pages 58–59 in [6]. This means that we only need to verify that

$$\sum_{|\lambda_j| \leq r_{n(\lambda_l)+m}} (1 - \varrho(\lambda_l, \lambda_j)) \leq m(\log 2)/2 + C$$

holds with C depending only on the constant of interpolation for Λ . Assuming Λ is an interpolating sequence and in view of (6), we may solve the problem $f_l(\lambda_l) = 2^{n(\lambda_l)/2}/\sqrt{1 - |\lambda_l|}$ and $f_l(\lambda_j) = 0$ for $j \neq l$ with uniform control of norms. Let us for simplicity set $r = r_{n(\lambda_l)+m}$. The function $\tilde{f}_l(z) = f_l(rz)$ has H^2 norm $\lesssim 2^{(n(\lambda_l)+m)/2}$. Now set

$$\varphi(z) = \frac{\lambda_j - rz}{r - \overline{\lambda_j}z}.$$

We apply Jensen's formula to $\tilde{f}_l \circ \varphi$ and get

$$(n(\lambda_j) \log 2 + \log \frac{1}{1 - |\lambda_l|})/2 + \sum_{j \neq l, |\lambda_j| < r} \log \frac{|\lambda_j - \lambda_l|}{|r - \overline{\lambda_l} \lambda_j / r|} = \int_0^{2\pi} \log |\tilde{f}_l \circ \varphi(e^{it})| \frac{dt}{2\pi}.$$

By the arithmetic-geometric mean inequality,

$$\int_0^{2\pi} \log |\tilde{f}_l \circ \varphi(e^{it})| \frac{dt}{2\pi} \leq \log \|\tilde{f}_l \circ \varphi\|_{H^2}.$$

Since the norm of the composition operator is bounded by an absolute constant times $(1 - |\lambda_l|)^{-1/2}$, we obtain

$$\int_0^{2\pi} \log |\tilde{f}_l \circ \varphi(e^{it})| dt \leq C + ((n(\lambda_j) + m) \log 2 + \log \frac{1}{1 - |\lambda_l|})/2.$$

What remains is to prove that

$$\sum_{j \neq l, |\lambda_j| < r} \log \frac{1}{|r - \overline{\lambda_l} \lambda_j / r|} \geq \sum_{j \neq l, |\lambda_j| < r} \log \frac{1}{|1 - \overline{\lambda_l} \lambda_j|} + C$$

for some constant C independent of l and m . This is a consequence of the inequality $|1 - z/a| \leq |1 - z|/a$ which holds whenever $|z| \leq a \leq 1$.

We turn to part (S). We obtain the condition $D_w^+(\Lambda) < \infty$ from the right inequality of (4), cf. Lemma 1. Following the reasoning on page 59 of [6], we find that it suffices to show that $D_w^-(\Lambda) \geq (\log 2)/2$. To prove this, we pick a point λ_l and look at the function

$$f_{l,m}(z) = \frac{1}{1 - \overline{\lambda_l} z} B_{l,m}(z),$$

where $B_{l,m}$ is the finite Blaschke product with zeros at the points λ_j for which $\lambda_j \neq \lambda_l$ and $|\lambda_j| \leq r_{n(\lambda_l)+m}$. By (5), we have

$$|f_{l,m}(\lambda_l)|^2 (1 - |\lambda_l|) 2^{-n(\lambda_l)} \lesssim \|f_{l,m}\|_w^2,$$

which implies that

$$e^{-2m(D^-(\Lambda)+o(1))} (1 - |\lambda_l|)^{-1} 2^{-n(\lambda_l)} \lesssim \|f_{l,m}\|_w^2$$

when $m \rightarrow \infty$ and λ_l is chosen appropriately. We now use the fact that the operators of multiplication by a single Blaschke factor are uniformly bounded below on A_w^2 . Thus applying the left inequality of (4) to the function

$$f_{l,m}(z) \frac{z - \lambda_l}{1 - \overline{\lambda_l} z}$$

and using (6), we get

$$\|f_{l,m}\|_w^2 \lesssim \sum_{|\lambda_j| > r_{n(\lambda_l)+m}} |f_{l,m}(\lambda_j)|^2 (1 - |\lambda_j|) 2^{-n(\lambda_j)}.$$

Now applying Theorem 1 and Carleson's embedding theorem for H^2 , we get

$$\|f_{l,m}\|_w^2 \lesssim (1 - |\lambda_l|)^{-1} 2^{-n(\lambda_l)-m},$$

and the desired estimate for $D_w^-(\Lambda)$ thus follows.

5. PROOF OF THE SUFFICIENCY OF THE CONDITIONS OF THEOREM 2

The main technical ingredient is the following lemma.

Lemma 2. *Let Λ be a separated sequence in the unit disk \mathbb{D} . (I) When $D_w^+(\Lambda) < (\log 2)/2$, there are $\varepsilon > 0$ and an analytic function $G(z)$ in \mathbb{D} with zero set $\Lambda' \supseteq \Lambda$ and*

$$|G(z)|^2 \simeq 2^{(1-\varepsilon)n(z)} \varrho^2(z, \Lambda').$$

(S) When $D_w^-(\Lambda) > (\log 2)/2$ and $D_w^+(\Lambda) < \infty$, there are $\varepsilon > 0$ and a meromorphic function $G(z)$ in \mathbb{D} with zero set Λ and pole set Λ' that is pseudohyperbolically separated from Λ and such that

$$|G(z)|^2 \simeq 2^{(1+\varepsilon)n(z)} \varrho^2(z, \Lambda) \varrho^{-2}(z, \Lambda').$$

Proof. In either case we begin by letting F be an analytic function having Λ as its zero set. We may write

$$\log |F(z)| = \sum_{n=0}^{\infty} (\log |B_n(z)| + h_n(z)),$$

where B_n is the Blaschke product with zeros at the λ_j in Ω_n and h_n is an appropriate harmonic function that makes the sum converge. The basic idea is to approximate the subharmonic function

$$U_j(z) = \sum_{n=m_j}^{mj+m-1} \log |B_n(z)|$$

by another subharmonic function $V_j(z)$ with Riesz measure supported by the circle $|z| = r_{mj}$; the point of this redistribution of the Riesz measure is that the latter measure is more easily atomized.

We choose m so large that either (I) $-U_j(z) \leq (1 - \varepsilon)m$ for $|z| = r_{mj}$, where $2\varepsilon = (\log 2)/2 - D_w^+(\Lambda)$ or (S) $-U_j(z) \geq (1 + \varepsilon)m$ for $|z| = r_{mj}$, where $2\varepsilon = D_w^-(\Lambda) - (\log 2)/2$. We claim that the function

$$V_j(z) = \frac{1}{\pi(1 - r_{mj}^2)} \int_0^{2\pi} \log \left| \frac{r_{mj}e^{it} - z}{1 - \bar{z}r_{mj}e^{it}} \right| U_j(r_{mj}e^{it}) dt$$

does the job in the sense that

$$\left| \sum_{j=1}^{\infty} (V_j(z) - U_j(z)) \right| \leq C$$

whenever $\varrho(\Lambda, z) \geq \delta > 0$ with C depending on δ . It is plain that we have

$$\left| \sum_{j: r_{mj} \leq |z|} (V_j(z) - U_j(z)) \right| \leq C$$

whenever $\varrho(\Lambda, z) \geq \delta > 0$. To deal with the case when $|z| < r_{mj}$, we note that then, by harmonicity, we may write

$$U_j(z) = \int_0^{2\pi} \frac{1 - |z|^2/r_{mj}^2}{|1 - \bar{z}e^{it}/r_{jm}|^2} U_j(r_{mj}e^{it}) \frac{dt}{2\pi}.$$

We approximate the logarithm in the integral defining V_j as

$$-\log \left| \frac{r_{mj}e^{it} - z}{1 - \bar{z}r_{mj}} \right| = \frac{1}{2} \frac{(1 - r_{mj}^2)(1 - |z|^2)}{|1 - \bar{z}r_{mj}e^{it}|^2} + O \left(\left[\frac{(1 - r_{mj}^2)(1 - |z|^2)}{|1 - \bar{z}r_{mj}e^{it}|^2} \right]^2 \right)$$

when $\varrho(z, r_{mj}e^{it}) \rightarrow 1$. Here the second order term causes no problem, so we only need to estimate the difference

$$D_r(z) = \frac{1 - |z|^2/r^2}{|1 - z/r|^2} - \frac{1 - |z|^2}{|1 - rz|^2}$$

when $|z| < r$. It suffices to observe that

$$D_r(z) = \frac{1}{r^2} \left(\frac{(1 - r^2)(1 - |z|^2)^2}{|1 - z/r|^2 |1 - rz|^2} - \frac{1 - r^2}{|1 - z/r|^2} \right)$$

because this identity implies that

$$\sum_{j: r_{mj} > |z|} |D_{r_{mj}}(z)| \lesssim 1.$$

We are now ready to construct the desired functions G in parts (I) and (S) respectively. To begin with, note that we have

$$2^{n(z)/2} \simeq \exp \left(\sum_{j=0}^{\infty} \frac{\log 2}{4\pi(1 - r_{mj})} \int_0^{2\pi} (\log |z - r_{mj}e^{it}| - \log r_{mj}) dt \right).$$

In other words, the function $n(z)$ can be approximated by a subharmonic function whose Riesz measure is concentrated on the circles $|z| = r_{mj}$ with density $\log 2 / (2\pi(1 - r_{mj}) \log 2)$ with respect to arc length measure on the respective circles. We now employ the counterpart of Lemma 5 on page 50 in [6]. In part (I), we thus produce an analytic function $H(z)$ by atomizing the Riesz measure of $(1 - \varepsilon)n(z) - V(z)$ for a sufficiently small $\varepsilon > 0$,

and then set $G = FH$. Similarly, in part (S), we construct an analytic function H by atomizing the Riesz measure of $V(z) - (1 + \varepsilon)n(z)$ for a sufficiently small $\varepsilon > 0$, and then set $G = F/H$. It is plain that the proof in [6] carries over to this situation. (The fact that the total mass of the Riesz measure we want to atomize over the circle $|z| = r_{mj}$ may be non-integer is of no significance. Just leave the “remainder” untouched; it corresponds to a bounded part of the subharmonic function.) \square

Proof of the sufficiency of the conditions in part (I) of Theorem 2. We want to solve the problem $f(\lambda_j) = a_j$ with f in A_w^2 when

$$\sum_j |a_j|^2 (1 - |\lambda_j|) 2^{-n(\lambda_j)} < \infty.$$

We will in fact construct a linear operator doing the job:

$$f(z) = \sum_j \frac{a_j}{G'(\lambda_j)} \frac{G(z)}{z - \lambda_j} \frac{1 - |\lambda_j|^2}{1 - \overline{\lambda_j}z},$$

just as formula (53) on page 53 of [6]. It follows from Lemma 2 that

$$|G'(\lambda_j)| \simeq 2^{(1-\varepsilon)n(\lambda_j)/2} (1 - |\lambda_j|)^{-1} \quad \text{and} \quad \frac{|G(z)|}{|z - \lambda_j|} \lesssim \frac{2^{(1-\varepsilon)n(z)/2}}{|1 - \overline{\lambda_j}z|}$$

so that

$$|f(z)| \lesssim 2^{(1-\varepsilon)n(z)/2} \sum_j |a_j| 2^{-(1-\varepsilon)n(\lambda_j)/2} \frac{(1 - |\lambda_j|)^2}{|1 - \overline{\lambda_j}z|^2}.$$

We write

$$h_n(z) = 2^{(1-\varepsilon)n(z)/2} \sum_{j: r_n \leq |\lambda_j| < r_{n+1}} |a_j| 2^{-(1-\varepsilon)n(\lambda_j)/2} \frac{(1 - |\lambda_j|)^2}{|1 - \overline{\lambda_j}z|^2}.$$

Thus we need to show that

$$\sum_{l=1}^{\infty} 2^{-l} \int_0^{2\pi} \left(\sum_{n=0}^{\infty} h_n(r_l e^{it}) \right)^2 dt \lesssim \sum_j |a_j|^2 (1 - |\lambda_j|) 2^{-n(\lambda_j)}.$$

We have

$$\sum_{l=1}^{\infty} 2^{-l} \int_0^{2\pi} \left(\sum_{n=0}^{\infty} h_n(r_l e^{it}) \right)^2 dt \leq I_1 + I_2,$$

where

$$I_1 = \sum_{l=1}^{\infty} 2^{-l+1} \int_0^{2\pi} \left(\sum_{n < l} h_n(r_l e^{it}) \right)^2 dt \quad \text{and} \quad I_2 = \sum_{l=1}^{\infty} 2^{-l+1} \int_0^{2\pi} \left(\sum_{n \geq l} h_n(r_l e^{it}) \right)^2 dt.$$

We compute the L^2 integral in I_1 by duality. Noting that $h_n(z)$ is a weighted sum of Poisson kernels and using the Carleson measure condition for Poisson integrals of L^2 functions along with the Cauchy–Schwarz inequality, we then get

$$I_1 \lesssim \sum_{l=1}^{\infty} 2^{-\varepsilon l} \left(\sum_{n < l} 2^{\varepsilon n/2} \left(\sum_{j: r_n \leq |\lambda_j| < r_{n+1}} |a_j|^2 (1 - |\lambda_j|) 2^{-n(\lambda_j)} \right)^{\frac{1}{2}} \right)^2.$$

By the Cauchy–Schwarz inequality, we get

$$I_1 \lesssim \sum_{l=1}^{\infty} 2^{-\varepsilon l/2} \sum_{n < l} 2^{\varepsilon n/2} \sum_{j: r_n \leq |\lambda_j| < r_{n+1}} |a_j|^2 (1 - |\lambda_j|) 2^{-n(\lambda_j)},$$

and the desired estimate is obtained if we change the order of summation. To deal with I_2 , we note that

$$\left(\sum_{n \geq l} h_n(z) \right)^2 \leq 2^{(1-\varepsilon)n(z)} \sum_{|\lambda_j| \geq r_l} |a_j|^2 2^{-(1-\varepsilon)n(\lambda_j)} \frac{(1 - |\lambda_j|)^2}{|1 - \overline{\lambda_j} z|^2} \sum_{|\lambda_k| \geq r_l} \frac{(1 - |\lambda_k|)^2}{|1 - \overline{\lambda_k} z|^2}.$$

Thus

$$\left(\sum_{n \geq l} h_n(r_l e^{it}) \right)^2 \lesssim 2^{(1-\varepsilon)l} \sum_{|\lambda_j| \geq r_l} |a_j|^2 2^{-(1-\varepsilon)n(\lambda_j)} \frac{(1 - |\lambda_j|)^2}{|1 - \overline{\lambda_j} r_l e^{ir_l}|^2}$$

from which it follows that

$$\int_0^{2\pi} \left(\sum_{n \geq l} h_n(r_l e^{it}) \right)^2 dt \lesssim 2^{(1-\varepsilon)l} \sum_{|\lambda_j| \geq r_l} |a_j|^2 2^{-(1-\varepsilon)n(\lambda_j)} \frac{(1 - |\lambda_j|)^2}{(1 - r_l)}.$$

Finally, we get

$$I_2 \lesssim \sum_j |a_j|^2 2^{-(1-\varepsilon)n(\lambda_j)} (1 - |\lambda_j|)^2 \sum_{r_l \leq |\lambda_j|} \frac{2^{-\varepsilon l}}{(1 - r_l)}.$$

In the latter sum, we can assume that ε is so small (if need be) that the terms grow exponentially, so that we may arrive at the desired estimate. \square

Proof of the sufficiency of the conditions in part (S) of Theorem 2. We start from the formula

$$f(z) = \sum_j \frac{f(\lambda_j)}{G'(\lambda_j)} \frac{G(z)}{z - z_j} \frac{1 - |z|^2}{1 - \overline{z} \lambda_j},$$

which holds for every function in A_w^2 , cf. formula (54) on page 53 of [6]. Here G is the meromorphic function of Lemma 2. We get that

$$|f(z)| \lesssim 2^{(1+\varepsilon)n(z)/2} (1 - |z|) \sum_j |f(\lambda_j)| 2^{-(1+\varepsilon)n(\lambda_j)/2} \frac{(1 - |\lambda_j|)}{|1 - \overline{\lambda_j} z|^2}.$$

We write

$$g_n(z) = 2^{(1+\varepsilon)n(z)/2}(1-|z|) \sum_{j: r_n \leq |\lambda_j| < r_{n+1}} |f(\lambda_j)| 2^{-(1+\varepsilon)n(\lambda_j)/2} \frac{(1-|\lambda_j|)}{|1-\overline{\lambda_j}z|^2}.$$

Thus we need to show that

$$\sum_{l=1}^{\infty} 2^{-l} \int_0^{2\pi} \left(\sum_{n=0}^{\infty} g_n(r_l e^{it}) \right)^2 dt \lesssim \sum_j |f(\lambda_j)|^2 (1-|\lambda_j|) 2^{-n(\lambda_j)}.$$

We write

$$\sum_{l=1}^{\infty} 2^{-l} \int_0^{2\pi} \left(\sum_{n=0}^{\infty} g_n(r_l e^{it}) \right)^2 dt \leq J_1 + J_2,$$

where

$$J_1 = \sum_{l=1}^{\infty} 2^{-l+1} \int_0^{2\pi} \left(\sum_{n < l} g_n(r_l e^{it}) \right)^2 dt \quad \text{and} \quad J_2 = \sum_{l=1}^{\infty} 2^{-l+1} \int_0^{2\pi} \left(\sum_{n \geq l} g_n(r_l e^{it}) \right)^2 dt.$$

We compute the L^2 integral in J_1 by duality. Using the Carleson measure condition and the Cauchy–Schwarz inequality, we get

$$J_1 \lesssim \sum_{l=1}^{\infty} (1-r_l)^{\frac{1}{2}} 2^{\varepsilon l} \sum_{n < l} \frac{2^{-\varepsilon n}}{(1-r_n)^{\frac{1}{2}}} \sum_{j: r_n \leq |\lambda_j| < r_{n+1}} |f(\lambda_j)|^2 (1-|\lambda_j|) 2^{-n(\lambda_j)}.$$

Changing the order of summation, we get the desired result. (We need to assume, if need be, that ε is so small that $2^{\varepsilon l}(1-r_l)^{\frac{1}{2}}$ decays exponentially.) To deal with J_2 , we note that

$$\left(\sum_{n \geq l} g_n(z) \right)^2 \leq 2^{(1+\varepsilon)l} (1-|z|)^2 \sum_{|\lambda_j| \geq r_l} |f(\lambda_j)|^2 2^{-(1+\frac{1}{2}\varepsilon)n(\lambda_j)} \frac{(1-|\lambda_j|)}{|1-\overline{\lambda_j}z|^2} \sum_{|\lambda_k| \geq r_l} \frac{(1-|\lambda_k|)}{|1-\overline{\lambda_k}z|^2} 2^{-\frac{1}{2}\varepsilon n(\lambda_k)}.$$

Applying the Carleson measure condition to the sum to the right, we get

$$\left(\sum_{n \geq l} g_n(r_l e^{it}) \right)^2 \leq 2^{l(1+\frac{1}{2}\varepsilon)} \sum_{|\lambda_j| \geq r_l} |f(\lambda_j)|^2 (1-|\lambda_j|) 2^{-(1+\frac{1}{2}\varepsilon)n(\lambda_j)} \frac{(1-r_l)}{|1-\overline{\lambda_j}r_l e^{it}|^2}$$

from which it follows that

$$\int_0^{2\pi} \left(\sum_{n \geq l} g_n(r_l e^{it}) \right)^2 dt \lesssim 2^{\varepsilon l/2} \sum_{|\lambda_j| \geq r_l} |f(\lambda_j)|^2 (1-|\lambda_j|) 2^{-(1+\frac{1}{2}\varepsilon)n(\lambda_j)}.$$

We now sum over l and get the desired result by changing the order of summation. \square

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